

WEIGHTED COMPOSITION OPERATORS ON SPACES OF ANALYTIC VECTOR-VALUED LIPSCHITZ FUNCTIONS

K. ESMAEILI,¹

ABSTRACT. Let φ be an analytic self-map of \mathbb{D} and ψ be an analytic operator-valued function on \mathbb{D} , where \mathbb{D} is the unit disk. We provide necessary and sufficient conditions for the boundedness and compactness of weighted composition operators $W_{\Psi, \phi} : f \mapsto \Psi(f \circ \phi)$ on $\text{Lip}_A(\mathbb{D}, X, \alpha)$ and $\text{lip}_A(\mathbb{D}, X, \alpha)$, the spaces of analytic X -valued Lipschitz functions f , where X is a complex Banach space and $\alpha \in (0, 1]$.

1. INTRODUCTION AND PRELIMINARIES

Let (S, d) be a metric space, X be a Banach space and $\alpha \in (0, 1]$. The space of all functions $f : S \rightarrow X$ for which

$$p_\alpha(f) = \sup \left\{ \frac{\|f(s_1) - f(s_2)\|}{d^\alpha(s_1, s_2)} : s_1, s_2 \in S, s_1 \neq s_2 \right\} < \infty,$$

is denoted by $\text{Lip}_\alpha(S, X)$. The subspace of functions f for which

$$\lim_{d(s_1, s_2) \rightarrow 0} \frac{\|f(s_1) - f(s_2)\|}{d^\alpha(s_1, s_2)} = 0,$$

is denoted by $\text{lip}_\alpha(S, X)$. The spaces $\text{Lip}_\alpha(S, X)$ and $\text{lip}_\alpha(S, X)$ equipped with norm $\|f\|_\alpha = \|f\|_S + p_\alpha(f)$ are Banach spaces, where,

$$\|f\|_S = \sup_{s \in S} \|f(s)\|.$$

These spaces are called X -valued Lipschitz spaces. The spaces $\text{Lip}_\alpha(S, X)$ and $\text{lip}_\alpha(S, X)$ were studied by Johnson for the first time [6].

For a Banach space X , let $H(\mathbb{D}, X)$ be the space of all analytic X -valued functions on the open unit disc \mathbb{D} and $A(\overline{\mathbb{D}}, X)$ be the Banach space of all continuous functions $f : \overline{\mathbb{D}} \rightarrow X$ which are analytic on \mathbb{D} . For $\alpha \in (0, 1]$ we define the spaces

$$\Lambda_\alpha(X) = \text{Lip}_\alpha(\mathbb{D}, X) \cap H(\mathbb{D}, X)$$

and

$$\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) = \text{Lip}_\alpha(\overline{\mathbb{D}}, X) \cap A(\overline{\mathbb{D}}, X).$$

Clearly, $\Lambda_\alpha(X)$ and $\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$, equipped with Lipschitz norm $\|\cdot\|_\alpha$, are Banach spaces and the spaces

$$\Lambda_\alpha^0(X) = \text{lip}_\alpha(\mathbb{D}, X) \cap H(\mathbb{D}, X)$$

2010 *Mathematics Subject Classification.* Primary 46E40; Secondary 47A56, 47B33.

Key words and phrases. Analytic vector-valued Lipschitz functions; vector-valued Bloch spaces; weighted composition operators; compact operators.

and

$$\text{lip}_A(\overline{\mathbb{D}}, X, \alpha) = \text{lip}_\alpha(\overline{\mathbb{D}}, X) \cap A(\overline{\mathbb{D}}, X)$$

are Banach subspaces of $\Lambda_\alpha(X)$ and $\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$, respectively.

We consider the weighted Banach space

$$H_\nu^\infty(X) = \{f \in H(\mathbb{D}, X) : \|f\|_\nu = \sup_{z \in \mathbb{D}} \nu(z) \|f(z)\| < \infty\}$$

endowed with norm $\|\cdot\|_\nu$, where $\nu : \mathbb{D} \rightarrow (0, +\infty)$ is a bounded continuous weight function.

For a positive real number α , $B_\alpha(X)$ denotes the X -valued Bloch type space of all analytic functions $f : \mathbb{D} \rightarrow X$ satisfying

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|f'(z)\| < \infty.$$

The space $B_\alpha(X)$ endowed with norm

$$\|f\|_{B_\alpha} = \|f(0)\| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|f'(z)\|, \quad (f \in B_\alpha(X)),$$

is a Banach space.

In the case $X = \mathbb{C}$, we omit X in the notation.

For Banach spaces X and Y , by $L(X, Y)$ ($K(X, Y)$), we mean the Banach space of all bounded (compact) linear operators from X to Y . Let $S(\overline{\mathbb{D}}, X)$ and $S(\overline{\mathbb{D}}, Y)$ be subspaces of $A(\overline{\mathbb{D}}, X)$ and $A(\overline{\mathbb{D}}, Y)$, respectively. Let $\phi \in A(\overline{\mathbb{D}})$ be a nonconstant self map of $\overline{\mathbb{D}}$ and $\Psi : \overline{\mathbb{D}} \rightarrow L(X, Y)$ be a continuous operator-valued function analytic on \mathbb{D} . Then the weighted composition operator $W_{\Psi, \phi}$ from $S(\overline{\mathbb{D}}, X)$ to $S(\overline{\mathbb{D}}, Y)$ is defined to be the linear operator of the form

$$W_{\Psi, \phi}(f)(z) = \psi(z)(f(\phi(z))), \quad (f \in S(\overline{\mathbb{D}}, X), \quad z \in \overline{\mathbb{D}}).$$

For simplicity of notation, we write Ψ_z instead of $\Psi(z)$.

Note that if Ψ_z is the identity map on X for every $z \in \overline{\mathbb{D}}$, then $W_{\Psi, \phi}$ is the composition operator on $S(\overline{\mathbb{D}}, X)$. In the scalar case, a weighted composition operator is a composition operator followed by a multiplier.

There is recent interest into properties of the composition operators and weighted composition operators between Banach spaces of vector-valued functions. For instance, the weakly compact composition operators on Hardy spaces, weighted Bergman spaces, Bloch spaces and BMOA, in the vector-valued case, have been characterized in [2, 8, 9, 11, 12]. Weighted composition operators between vector-valued Lipschitz spaces and weighted Banach spaces of vector-valued analytic functions have been studied in [5, 10]. Also, composition operators on analytic Lipschitz spaces in the scalar-valued case have been investigated in [1, 13]. The present study is aimed at finding some necessary and sufficient conditions for boundedness and compactness of weighted composition operators on the spaces of analytic X -valued Lipschitz functions.

The rest of this paper is designed as follows. In section 2 we shall characterize bounded and compact weighted composition operators between analytic vector-valued Lipschitz spaces. Section 3 is devoted to discussing this kinds of operators between analytic vector-valued little Lipschitz spaces.

2. WEIGHTED COMPOSITION OPERATORS ON $\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$ AND $\Lambda_\alpha(X)$

In this section, we provide necessary and sufficient conditions for weighted composition operators between analytic vector-valued Lipschitz spaces to be bounded and compact. In what follows, we will assume that X and Y are Banach spaces.

We begin with some elementary properties of analytic vector-valued Lipschitz spaces. The best reference here is [2]. Let E be a Banach subspace of $H(\mathbb{D})$ which contains constant functions and its closed unit ball $U(E)$ is compact for the compact open topology. Then the space

$${}^*E := \{u \in E^* : u|_{U(E)} \text{ is co-continuous}\},$$

endowed with the norm induced by E^* , is a Banach space and the evaluation map $f \mapsto [u \mapsto u(f)]$ from E into $({}^*E)^*$, is an isometric isomorphism. In particular, *E is a predual of E . Furthermore, the vector-valued space

$$E[X] := \{f \in H(\mathbb{D}, X) : x^* \circ f \in E, \quad x^* \in X^*\},$$

by the norm $\|f\|_{E[X]} = \sup_{\|x^*\| \leq 1} \|x^* \circ f\|$ is a Banach space.

Bonet, et al. in [2, Lemma 10] argued that the map $\Delta : \mathbb{D} \rightarrow {}^*E$, $\Delta(z) = \delta_z$, where δ_z is the evaluation map on E , is analytic and the linear operator $\chi : L({}^*E, X) \rightarrow E[X]$, $\chi(T) = T \circ \Delta$ is bounded. Defining $\psi(g)(u) : X^* \rightarrow \mathbb{C}$ by $\psi(g)(u)(x^*) = u(x^* \circ g)$ for $g \in E[X]$ and $u \in {}^*E$, they showed that $\psi(g) \in L({}^*E, X^{**})$ and $\psi(g)(\delta_z) \in L({}^*E, X)$. Besides, using operators χ and ψ , they deduced that the space $E[X]$ is isomorphic to $L({}^*E, X)$.

In the following proposition we use the above mentioned result for the spaces $\Lambda_\alpha[X]$ and $B_{1-\alpha}[X]$ to show that the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{B_{1-\alpha}(X)}$ are equivalent, whenever $\alpha \in (0, 1)$.

Proposition 2.1. *Let $\alpha \in (0, 1)$. Then $\Lambda_\alpha(X) = B_{1-\alpha}(X)$ and the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{B_{1-\alpha}}$ are equivalent.*

Proof. Using Hardy-Littlewood theorem for $\alpha \in (0, 1)$, we can see that $\Lambda_\alpha = B_{1-\alpha}$ and $\|\cdot\|_\alpha \asymp \|\cdot\|_{B_{1-\alpha}}$ (i.e. for some positive constants a and b , $a\|\cdot\|_\alpha \leq \|\cdot\|_{B_{1-\alpha}} \leq b\|\cdot\|_\alpha$). Hence, ${}^*\Lambda_\alpha = {}^*B_{1-\alpha}$, where ${}^*\Lambda_\alpha$ and ${}^*B_{1-\alpha}$ are the preduals of Λ_α and $B_{1-\alpha}$ mentioned above, respectively. Thus $\Lambda_\alpha[X] = B_{1-\alpha}[X]$ and the linear operators

$$\text{id} : \Lambda_\alpha[X] \xrightarrow{\psi} L({}^*\Lambda_\alpha, X) = L({}^*B_{1-\alpha}, X) \xrightarrow{\chi} B_{1-\alpha}[X]$$

and

$$\text{id} : B_{1-\alpha}[X] \xrightarrow{\psi} L({}^*B_{1-\alpha}, X) = L({}^*\Lambda_\alpha, X) \xrightarrow{\chi} \Lambda_\alpha[X]$$

are bounded. This shows that $\|\cdot\|_{\Lambda_\alpha[X]} \asymp \|\cdot\|_{B_{1-\alpha}[X]}$. Since $\Lambda_\alpha(X) = \Lambda_\alpha[X]$ and $B_{1-\alpha}(X) = B_{1-\alpha}[X]$, we conclude that

$$\|\cdot\|_\alpha = \|\cdot\|_{\Lambda_\alpha[X]} \asymp \|\cdot\|_{B_{1-\alpha}[X]} = \|\cdot\|_{B_{1-\alpha}}.$$

□

From Proposition 2.1 we deduce that for $\alpha \in (0, 1)$ the norm

$$\|f\|_{\Lambda_\alpha(X)} = \|f(0)\| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{(1-\alpha)} \|f'(z)\|, \quad (f \in \Lambda_\alpha(X))$$

defines an equivalent norm on $\Lambda_\alpha(X)$. Hereafter we use this norm for $\Lambda_\alpha(X)$ and $\Lambda_\alpha^0(X)$, whenever $\alpha \in (0, 1)$.

In [13], Mahyar and Sanatpour proved that every function $f \in \text{Lip}_\alpha(\mathbb{D})$ has a unique continuous extension $E(f)$ to $\overline{\mathbb{D}}$, such that $E(f) \in \text{Lip}_\alpha(\overline{\mathbb{D}})$. In fact, for every $w \in \partial\mathbb{D}$ they defined $E(f)(w) = \lim_{n \rightarrow \infty} f(z_n)$, where $\{z_n\}$ is any sequence in \mathbb{D} converging to w . By the same method, one can see that every function $f \in \text{Lip}_\alpha(\mathbb{D}, X)$ has a unique continuous extension $E(f)$ to $\overline{\mathbb{D}}$ such that $E(f) \in \text{Lip}_\alpha(\overline{\mathbb{D}}, X)$.

Furthermore, as in the proof of [13, Proposition 2.1], it can be seen that the mapping $f \mapsto E(f)$ is a homeomorphism from $(\Lambda_\alpha(X), \|\cdot\|_{\Lambda_\alpha(X)})$ onto $(\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha), \|\cdot\|_\alpha)$. Thus we can modify the problem of boundedness and compactness of the weighted composition operators $W_{\Psi, \phi} : \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \rightarrow \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta)$, to the problem of boundedness and compactness of the weighted composition operators $W_{\psi, \varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$, where $\psi = \Psi|_{\mathbb{D}}$ and $\varphi = \phi|_{\mathbb{D}} : \mathbb{D} \rightarrow \mathbb{D}$.

Proposition 2.2. *Let $\Psi \in \text{Lip}_A(\overline{\mathbb{D}}, L(X, Y), \beta)$ and $\Phi \in A(\overline{\mathbb{D}})$ be a self map of $\overline{\mathbb{D}}$. Then $W_{\Psi, \phi} : \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \rightarrow \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta)$ is bounded (compact) if and only if $W_{\psi, \varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ is bounded (compact), where $\psi = \Psi|_{\mathbb{D}}$ and $\varphi = \phi|_{\mathbb{D}} : \mathbb{D} \rightarrow \mathbb{D}$.*

Proof. Let R be the restriction map from $\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$ into $\Lambda_\alpha(X)$. Clearly, R is a bounded linear operator. Let $z \in \partial\mathbb{D}$ and $\{z_n\}$ be any sequence in \mathbb{D} converging to z . An easy computation shows that for every $f \in \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$,

$$W_{\Psi, \phi}(f)(z) = \lim_{n \rightarrow \infty} \psi_{z_n} f(\varphi(z_n)).$$

On the other hand,

$$(E \circ W_{\psi, \varphi} \circ R)(f)(z) = \lim_{n \rightarrow \infty} W_{\psi, \varphi}(R(f))(z_n) = \lim_{n \rightarrow \infty} \psi_{z_n} f(\varphi(z_n))$$

holds for every $f \in \Lambda_\alpha(X)$. Thus $E \circ W_{\psi, \varphi} \circ R = W_{\Psi, \phi}$ and the diagram

$$\begin{array}{ccc} \Lambda_\alpha(X) & \xrightarrow{W_{\psi, \varphi}} & \Lambda_\beta(Y) \\ \uparrow R & & \downarrow E \\ \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) & \xrightarrow{W_{\Psi, \phi}} & \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta) \end{array}$$

is commutative.

Likewise, $W_{\psi, \varphi} = R \circ W_{\Psi, \phi} \circ E$ and the diagram

$$\begin{array}{ccc} \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) & \xrightarrow{W_{\Psi, \phi}} & \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta) \\ \uparrow E & & \downarrow R \\ \Lambda_\alpha(X) & \xrightarrow{W_{\psi, \varphi}} & \Lambda_\beta(Y) \end{array}$$

is commutative and the proof is complete. \square

For every $x \in X$ and every scalar-valued function $f \in \Lambda_\alpha$, the function $f_x(z) = f(z)x$, $z \in \mathbb{D}$ belongs to $\Lambda_\alpha(X)$ and $\|f_x\|_{\Lambda_\alpha(X)} = \|f\|_{\Lambda_\alpha} \|x\|$. Moreover,

$$(W_{\psi, \varphi}(f_x))'(z) = \varphi'(z) f'(\varphi(z)) \psi_z(x) + f(\varphi(z)) \psi'_z(x).$$

In particular, for each $x \in X$ the constant function 1_x exists in $\Lambda_\alpha(X)$ and the Banach space X can be considered as a subspace of $\Lambda_\alpha(X)$.

For every vector-valued function $f \in H(\mathbb{D}, X)$ and every $z \in \mathbb{D}$ we have

$$(W_{\psi,\varphi}(f))'(z) = \varphi'(z)\psi_z(f'(\varphi(z))) + \psi'_z(f(\varphi(z))).$$

Hence, $DW_{\psi,\varphi} = W_{\varphi'\psi,\varphi} \circ D + W_{\psi',\varphi}$, where D is the derivation operator. By [10, Theorem 2.1], for $0 < \alpha < 1$ and $0 < \beta \leq 1$ we have

$$\|W_{\psi,\varphi} : H_{\nu_\alpha}^\infty(X) \rightarrow H_{\nu_\beta}^\infty(Y)\| \asymp \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} \|\psi_z\|, \quad (2.1)$$

where ν_α is the standard weight $\nu_\alpha(z) = (1 - |z|^2)^\alpha$.

By the same method, one can see that $W_{\psi,\varphi} : H^\infty(X) \rightarrow H_{\nu_\beta}^\infty(Y)$ is bounded if and only if $\psi \in H_{\nu_\beta}^\infty(L(X, Y))$. Moreover, $\|W_{\psi,\varphi} : H^\infty(X) \rightarrow H_{\nu_\beta}^\infty(Y)\| \asymp \|\psi\|_{H_{\nu_\beta}^\infty}$.

The following theorem characterizes the bounded weighted composition operators between analytic vector-valued Lipschitz spaces.

Theorem 2.3. *For $0 < \alpha < 1$ the operator $W_{\psi,\varphi}$ maps $\Lambda_\alpha(X)$ boundedly into $\Lambda_\beta(Y)$ if and only if $\psi \in \Lambda_\beta(L(X, Y))$ and*

$$q_{\alpha,\beta} = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| < \infty.$$

Moreover,

$$\|W_{\psi,\varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)\| \asymp \max\{q_{\alpha,\beta}, \|\psi\|_{\Lambda_\beta(L(X,Y))}\}. \quad (2.2)$$

Proof. The proof of the first part is a straightforward modification of that of [7, Theorem 2.1]. We prove that in the case $W_{\psi,\varphi}$ is bounded, the relation (2.2) holds. Let $W_{\psi,\varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ be bounded. Using $DW_{\psi,\varphi} = W_{\varphi'\psi,\varphi} \circ D + W_{\psi',\varphi}$, one can easily show that the operators $W_{\varphi'\psi,\varphi} : H_{\nu_{1-\alpha}}^\infty(X) \rightarrow H_{\nu_{1-\beta}}^\infty(Y)$ and $W_{\psi',\varphi} : \Lambda_\alpha(X) \rightarrow H_{\nu_{1-\beta}}^\infty(Y)$ are bounded and

$$\|W_{\psi',\varphi} : \Lambda_\alpha(X) \rightarrow H_{\nu_{1-\beta}}^\infty(Y)\| \leq \frac{2}{\alpha} \|\psi\|_{\Lambda_\beta(L(X,Y))}.$$

Therefore,

$$\begin{aligned} \|W_{\psi,\varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)\| &\leq \|W_{\varphi'\psi,\varphi} : H_{\nu_{1-\alpha}}^\infty(X) \rightarrow H_{\nu_{1-\beta}}^\infty(Y)\| \\ &\quad + \|W_{\psi',\varphi} : \Lambda_\alpha(X) \rightarrow H_{\nu_{1-\beta}}^\infty(Y)\|. \end{aligned}$$

From relation (2.1), for some positive constant C we have

$$\|W_{\psi,\varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)\| \leq C \max\{q_{\alpha,\beta}, \|\psi\|_{\Lambda_\beta(L(X,Y))}\}.$$

For the converse, since

$$\begin{aligned} (1 - |z|^2)^{1-\beta} \|\psi'_z\| &= \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|\psi'_z(x)\| \\ &= \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(1_x))'(z)\| \end{aligned}$$

holds for every $z \in \mathbb{D}$ and since $W_{\psi,\varphi}$ is bounded, we have

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|\psi'_z\| \leq \|W_{\psi,\varphi}\|.$$

For every nonzero $a \in \mathbb{D}$ and every $x \in X$ define

$$f_{a,x}(z) = \frac{1}{\bar{a}} \left(\frac{1 - |a|^2}{(1 - \bar{a}z)^{1-\alpha}} - (1 - \bar{a}z)^\alpha \right) x.$$

It could be shown that $\{f_{a,x} : a \neq 0, \|x\| \leq 1\}$ is a bounded subset of $\Lambda_\alpha(X)$. Moreover, $f_{a,x}(a) = 0$ and $f'_a(a) = \frac{x}{(1-|a|^2)^{1-\alpha}}$. Then for some positive constant C we have

$$q_{\alpha,\beta} = \sup_{\|x\| \leq 1} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f_{\varphi(z),x}))'(z)\| \leq C \|W_{\psi,\varphi}\|,$$

which completes the proof. \square

Theorem 2.4. *The operator $W_{\psi,\varphi} : \Lambda_1(X) \rightarrow \Lambda_\beta(Y)$ is well defined and bounded if and only if $\psi \in \Lambda_\beta(L(X, Y))$ and $\varphi'\psi \in H_{\nu_1-\beta}^\infty(L(X, Y))$. Furthermore,*

$$\|W_{\psi,\varphi} : \Lambda_1(X) \rightarrow \Lambda_\beta(Y)\| \asymp \max\{\|\varphi'\psi\|_{H_{\nu_1-\beta}^\infty}, \|\psi\|_{\Lambda_\beta}\}.$$

Proof. A simple computation gives that $f \in \Lambda_1(X)$ if and only if $f' \in H^\infty(X)$ and $\|f\|_1 \asymp \|f\|_{\mathbb{D}} + \|f'\|_{\mathbb{D}}$.

Let $W_{\psi,\varphi}$ be bounded. Defining $f_{a,x}(z) = (z - a)x$, for any $a, z \in \mathbb{D}$ and $x \in X$, one can see that $\{f_{a,x} : a \in \mathbb{D}, \|x\| \leq 1\}$ is a bounded sequence in $\Lambda_1(X)$. Since $W_{\psi,\varphi}$ is bounded, for every $z \in \mathbb{D}$,

$$\begin{aligned} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| &= \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z x\| \\ &= \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f_{\varphi(z),x}))'(z)\| \\ &\leq \sup_{\|x\| \leq 1} \|W_{\psi,\varphi}\| \|f_{\varphi(z),x}\|_{\Lambda_1(X)} < 3 \|W_{\psi,\varphi}\|. \end{aligned}$$

Thus $\varphi'\psi \in H_{\nu_1-\beta}^\infty(L(X, Y))$. The rest of the proof runs as the proof of Theorem 2.3. \square

The next lemma shows that the compact open topology on $\Lambda_\alpha^0(X)$ is stronger than the weak topology.

Lemma 2.5. *Let $\alpha \in (0, 1)$ and $\{f_n\}$ be a bounded sequence in $\Lambda_\alpha^0(X)$ converging to zero uniformly on compact subsets of \mathbb{D} . Then $\{f_n\}$ converges weakly to zero.*

Proof. For every $f \in \Lambda_\alpha^0(X)$, consider the function $\tilde{f}(z) = (1 - |z|^2)^{1-\alpha} f'(z)$ and set $\tilde{\Lambda} = \{\tilde{f} : f \in \Lambda_\alpha^0(X)\}$. Clearly, $\tilde{\Lambda}$ is a subspace of $C_o(\mathbb{D}, X)$. Let T be a bounded linear functional on $\Lambda_\alpha^0(X)$. By Hahn- Banach theorem, for some measure $\mu \in M(\mathbb{D}, X^*)$ we have $Tf = \int_{\mathbb{D}} \tilde{f} d\mu$, for every $f \in \Lambda_\alpha^0(X)$ (see [4, Corollary 2, p. 387]). Without loss of generality we assume that $\|f_n\|_{\Lambda_\alpha(X)} \leq 1$. Fixing $\epsilon > 0$, let $\{r_m\}$ be an increasing sequence in $(0, 1)$ converging to 1 and $D_m = \{z \in \mathbb{D} : |z| \leq r_m\}$. Then $\mathbb{D} = \cup_{m=1}^\infty D_m$ and $|\mu|(\mathbb{D} \setminus D_m) < \frac{\epsilon}{2}$ for some m . Since $\{f_n\}$ converges to zero uniformly on compact subsets of \mathbb{D} , we deduce that

$\sup_{z \in D_m} \|\tilde{f}_n(z)\| < \frac{\epsilon}{2\|\mu\|}$ for n sufficiently large. Therefore

$$\begin{aligned} |T(f_n)| &\leq \left| \int_{\mathbb{D} \setminus D_m} \tilde{f}_n d\mu \right| + \left| \int_{D_m} \tilde{f}_n d\mu \right| \\ &\leq \int_{\mathbb{D} \setminus D_m} \|\tilde{f}_n(z)\| |d\mu|(z) + \int_{D_m} \|\tilde{f}_n(z)\| |d\mu|(z) \\ &\leq |\mu|(\mathbb{D} \setminus D_m) + |\mu|(D_m) \frac{\epsilon}{2\|\mu\|} < \epsilon, \end{aligned}$$

which shows that $\lim_{n \rightarrow \infty} T(f_n) = 0$. Thus $\{f_n\}$ converges weakly to zero as desired. \square

In the next theorem we provide a necessary and sufficient condition for weighted composition operator $W_{\psi, \varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ to be compact, whenever $\alpha \in (0, 1)$. We use the idea of [10] and define $T_\psi : X \rightarrow \Lambda_\beta(Y)$, by $T_\psi(x)(z) = \psi_z(x)$. In the case $W_{\psi, \varphi}$ is bounded, T_ψ is a bounded linear operator and

$$\|T_\psi : X \rightarrow \Lambda_\beta(Y)\| \leq \|W_{\psi, \varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)\|.$$

For $n \in \mathbb{N}$, we define $L_n : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ by $L_n(f) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} z^k$, for every $f \in \Lambda_\alpha(X)$, where $\sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} z^k$ is the Taylor expansion of f . One can easily show that for every $f \in \Lambda_\alpha^0(X)$, $\|L_n f - f\|_{\Lambda_\alpha(X)} \rightarrow 0$ as $n \rightarrow \infty$.

For $r \in (0, 1)$ we define the linear operator $K_r : \Lambda_\alpha(X) \rightarrow \Lambda_\alpha(X)$, $K_r(f)(z) = f(rz)$ for every $f \in \Lambda_\alpha(X)$ and $z \in \mathbb{D}$. Clearly, K_r is a bounded linear operator from $\Lambda_\alpha(X)$ into $\Lambda_\alpha(X)$ and for every $f \in \Lambda_\alpha(X)$, $\|K_r f - f\|_{\Lambda_\alpha(X)} \rightarrow 0$ as $r \rightarrow 1^-$.

Theorem 2.6. *Let $0 < \alpha < 1$ and $W_{\psi, \varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ be a bounded weighted composition operator. Then $W_{\psi, \varphi}$ is compact if and only if T_ψ is compact and*

$$\limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| = 0. \quad (2.3)$$

Proof. Let $W_{\psi, \varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ be compact. Considering the bounded operator $T : X \rightarrow \Lambda_\alpha(X)$, $Tx = 1_x$, we get $T_\psi = W_{\psi, \varphi} \circ T$ is compact. If (2.3) does not hold, one can find a sequence $\{z_n\}$ of \mathbb{D} such that $|\varphi(z_n)| > \frac{1}{2}$, $|\varphi(z_n)| \rightarrow 1$ and

$$\lim_{n \rightarrow \infty} \frac{(1 - |z_n|^2)^{1-\beta}}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} |\varphi'(z_n)| \|\psi_{z_n}\| > 0.$$

For every n , define

$$f_n(z) = \frac{1}{\varphi(z_n)} \left(\frac{(1 - |\varphi(z_n)|^2)^2}{(1 - \overline{\varphi(z_n)}z)^{2-\alpha}} - \frac{1 - |\varphi(z_n)|^2}{(1 - \overline{\varphi(z_n)}z)^{1-\alpha}} \right) x_n$$

where $\{x_n\}$ is a sequence in X , for which $\|x_n\| \leq 1$ and $\frac{n}{n+1} \|\psi_{z_n}\| < \|\psi_{z_n}(x_n)\|$. Clearly, $\{f_n\}$ is a bounded sequence in $\Lambda_\alpha^0(X)$ converging to zero on compact subsets of \mathbb{D} . Moreover, $f_n(\varphi(z_n)) = 0$ and $f'_n(\varphi(z_n)) = \frac{1}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} x_n$. Since

$W_{\psi,\varphi}$ is compact, from lemma 2.5 we deduce that $W_{\psi,\varphi}(f_n) \rightarrow 0$, as $n \rightarrow \infty$. But

$$\begin{aligned} \|W_{\psi,\varphi}(f_n)\|_{\Lambda_\beta(Y)} &\geq (1 - |z_n|^2)^{1-\beta} \|\varphi'(z_n) f'_n(\varphi(z_n)) \psi_{z_n}(x_n) + f_n(\varphi(z_n)) \psi'_{z_n}(x_n)\| \\ &= \frac{(1 - |z_n|^2)^{1-\beta}}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} |\varphi'(z_n)| \|\psi_{z_n}(x_n)\| \\ &> \frac{(1 - |z_n|^2)^{1-\beta}}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} |\varphi'(z_n)| \|\psi_{z_n}\| \frac{n}{n+1}, \end{aligned}$$

implies that $\lim_{n \rightarrow \infty} \|W_{\psi,\varphi}(f_n)\|_{\Lambda_\beta(Y)} > 0$, which is impossible.

Conversely, let T_ψ be compact and (2.3) holds. Consider the operator $q_k : \Lambda_\alpha(X) \rightarrow X$, $f \mapsto \frac{f^{(k)}(0)}{k!}$. By Cauchy's integral theorem, for every $f \in \Lambda_\alpha(X)$ we have

$$\|q_k(f)\| \leq \frac{1}{2\pi} \oint_{|z|=\frac{1}{2}} \frac{\|f'(z)\|}{|z|^k} d|z| \leq \frac{2^k}{C} \|f\|_{\Lambda_\alpha(X)},$$

where C is a positive constant. Therefore q_k is a bounded linear operator. We show that $M_{\varphi^k} : \Lambda_\beta(Y) \rightarrow \Lambda_\beta(Y)$, $f \mapsto \varphi^k f$ is bounded. Since $W_{\psi,\varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ is bounded, from Theorem 2.3 we conclude that

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| < \infty.$$

Thus for every $f \in \Lambda_\beta(Y)$ we have

$$\begin{aligned} \|M_{\varphi^k}(f)\|_{\Lambda_\beta(Y)} &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|(\varphi^k f)'(z)\| \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|f'(z)\| + k \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|f(z)\| \\ &\leq \|f\|_{\Lambda_\beta(Y)} + k \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} (1 - |\varphi(z)|^2)^{1-\alpha} |\varphi'(z)| \|f(z)\| \\ &\leq (2 + kC) \|f\|_{\Lambda_\beta(Y)}, \end{aligned}$$

where C is a positive constant. This shows that M_{φ^k} is a bounded operator on $\Lambda_\beta(Y)$ and since T_ψ is compact, we deduce that $W_{\psi,\varphi} \circ L_n = \sum_{k=0}^n M_{\varphi^k} \circ T_\psi \circ q_k$ is a compact operator. Since $K_r f \in \Lambda_\alpha^0(X)$, we see that $\|K_r f - L_n(K_r f)\|_{\Lambda_\alpha(X)} \rightarrow 0$ as $n \rightarrow \infty$. Hence $\|W_{\psi,\varphi} \circ K_r - W_{\psi,\varphi} \circ L_n \circ K_r\| \rightarrow 0$ as $n \rightarrow \infty$, which shows that $W_{\psi,\varphi} \circ K_r$ is a compact operator. For completing the proof we show that $\limsup_{r \rightarrow 1^-} \|W_{\psi,\varphi} - W_{\psi,\varphi} \circ K_r\| = 0$. For this, fix $0 < \delta < 1$. For every $f \in \Lambda_\alpha(X)$

with $\|f\|_{\Lambda_\alpha(X)} \leq 1$ we have

$$\begin{aligned} & \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f - K_r f))'(z)\| \\ & \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| \|f'(\varphi(z)) - r f'(r\varphi(z))\| \\ & + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|\psi'_z\| \|f(\varphi(z)) - f(r\varphi(z))\| \\ & \leq \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| \|f'(\varphi(z)) - f'(r\varphi(z))\| \end{aligned} \quad (2.4)$$

$$+ \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| \|f'(r\varphi(z))\| (1 - r) \quad (2.5)$$

$$+ \sup_{|\varphi(z)| > \delta} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| \|(f - f_r)'(\varphi(z))\| \quad (2.6)$$

$$+ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|\psi'_z\| \|f(\varphi(z)) - f(r\varphi(z))\|. \quad (2.7)$$

By Cauchy's integral theorem, (2.4) is not bigger than $\frac{(1-r)}{(1-\delta)^{3-\alpha}}$ and converges to zero whenever $r \rightarrow 1^-$. Also,

$$(2.5) \leq \sup_{|\varphi(z)| \leq \delta} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| (1 - r) \rightarrow 0,$$

as $r \rightarrow 1^-$ and

$$(2.6) \leq 2 \sup_{|\varphi(z)| > \delta} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\|.$$

Moreover,

$$(2.6) \leq \sup_{z \in \mathbb{D}} \|\psi_z\| \|f\|_\alpha |\varphi(z)| (1 - r)^\alpha \rightarrow 0,$$

as $r \rightarrow 1^-$. Thus

$$\begin{aligned} \lim_{r \rightarrow 1^-} \|W_{\psi,\varphi} - W_{\psi,\varphi} \circ K_r\| &= \lim_{r \rightarrow 1^-} \sup_{\|f\| \leq 1} \|W_{\psi,\varphi}(f) - W_{\psi,\varphi} \circ K_r(f)\| \\ &\leq 2 \sup_{|\varphi(z)| > \delta} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\|. \end{aligned}$$

Letting $\delta \rightarrow 1$, we have

$$\lim_{r \rightarrow 1^-} \|W_{\psi,\varphi} - W_{\psi,\varphi} \circ K_r\| \leq 2 \lim_{\delta \rightarrow 1} \sup_{|\varphi(z)| > \delta} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| = 0,$$

which ensures that $W_{\psi,\varphi}$ is compact. \square

3. WEIGHTED COMPOSITION OPERATORS ON $\text{lip}_A(\overline{\mathbb{D}}, X, \alpha)$ AND $\Lambda_\alpha^0(X)$

In this section we characterize bounded and compact weighted composition operators on the spaces of analytic vector-valued little Lipschitz functions.

Ohno, et al. in [7, Theorem 4.1] showed that for $\alpha \in (0, 1)$, the operator $W_{\psi,\varphi} : \Lambda_\alpha^0 \rightarrow \Lambda_\beta^0$ is bounded if and only if $\psi \in \Lambda_\beta^0$, $W_{\psi,\varphi} : \Lambda_\alpha \rightarrow \Lambda_\beta$ is bounded and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{1-\beta} \varphi'(z) \psi_z = 0$. For the vector-valued case, we have weaker

results. That is, if $W_{\psi,\varphi} : \Lambda_\alpha^0(X) \rightarrow \Lambda_\beta^0(Y)$ is well defined and bounded, then $\psi \in \Lambda_\beta(\mathbb{D}, L(X, Y))$ and the point wise limit of $(1 - |z|^2)^{1-\beta} \varphi'(z) \psi_z$ is zero, whenever $|z| \rightarrow 1^-$.

By the next theorem, we provide some sufficient conditions for the weighted composition operator $W_{\psi,\varphi} : \Lambda_\alpha^0(X) \rightarrow \Lambda_\beta^0(Y)$ to be well defined and bounded.

Theorem 3.1. *Let $W_{\psi,\varphi} : \Lambda_\alpha(X) \rightarrow \Lambda_\beta(Y)$ be bounded, $\psi \in \Lambda_\beta^0(\mathbb{D}, L(X, Y))$ and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{1-\beta} \varphi'(z) \psi_z = 0$. Then $W_{\psi,\varphi} : \Lambda_\alpha^0(X) \rightarrow \Lambda_\beta^0(Y)$ is well defined and bounded.*

Proof. We just show that $W_{\psi,\varphi} : \Lambda_\alpha^0(X) \rightarrow \Lambda_\beta^0(Y)$ is well defined. The boundedness can be shown by means of the closed graph theorem. Let $f \in \Lambda_\alpha^0(X)$. Given $\epsilon > 0$, for some $r \in (0, 1)$ and for every $z \in \mathbb{D}$ with $r < |z| < 1$, we have

$$\begin{aligned} (1 - |z|^2)^{1-\beta} \|\psi'_z\| &< \frac{\alpha\epsilon}{4\|f\|_{\Lambda_\alpha(X)}}, \\ (1 - |z|^2)^{1-\alpha} \|f'(z)\| &< \frac{\epsilon}{4M}, \\ (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| &< \frac{\epsilon}{4L}, \end{aligned}$$

where

$$M = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| < \infty,$$

and

$$L = \max\left\{\sup_{|z| \leq r} \|f(z)\|, \sup_{|z| \leq r} \|f'(z)\|\right\} < \infty.$$

Fix $z \in \mathbb{D}$ with $r < |z| < 1$. If $r < |\varphi(z)| < 1$, then

$$\begin{aligned} &(1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z(f'(\varphi(z)))\| \\ &\leq \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} (1 - |\varphi(z)|^2)^{1-\alpha} |\varphi'(z)| \|\psi_z\| \|f'(\varphi(z))\| < \frac{\epsilon}{4}, \end{aligned}$$

and for $|\varphi(z)| \leq r$ we have

$$(1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z(f'(\varphi(z)))\| \leq L(1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z\| < \frac{\epsilon}{4}.$$

This shows that

$$\sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z(f'(\varphi(z)))\| < \frac{\epsilon}{2}.$$

It is easy to check that $\|f((\varphi(z)))\| \leq \frac{1}{\alpha} \|f\|_{\Lambda_\alpha(X)}$. Thus

$$\begin{aligned} \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f))'(z)\| &\leq \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z(f'(\varphi(z)))\| \\ &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \|\psi'_z(f(\varphi(z)))\| < \epsilon, \end{aligned}$$

which ensures that $W_{\psi,\varphi}(f) \in \Lambda_\beta^0(Y)$. □

For characterizing the compact weighted composition operators between analytic little Lipschitz spaces we need the next lemma.

Lemma 3.2. *A subset K of $\text{lip}_A(\overline{\mathbb{D}}, X, \alpha)$ is relatively compact if and only if*

- (i) K is bounded,
- (ii) $K(z) = \{f(z) : f \in K\}$ is relatively compact for every $z \in \overline{\mathbb{D}}$,
- (iii) $\lim_{\substack{|z| \rightarrow 1^- \\ z \in \mathbb{D}}} \sup_{f \in K} (1 - |z|^2)^{1-\alpha} \|f'(z)\| = 0$.

Proof. We begin by proving that (iii) is necessary. Suppose that K is relatively compact and let C be a positive constant such that $\|\cdot\|_{\Lambda_\alpha(X)} \leq C\|\cdot\|_\alpha$. Given $\epsilon > 0$, there are functions $f_1, f_2, \dots, f_n \in K$ such that for every $f \in K$ and for some $1 \leq j \leq n$, we have $\|f - f_j\|_\alpha < \frac{\epsilon}{2C}$ which ensures that

$$(1 - |z|^2)^{1-\alpha} \|f'(z)\| \leq (1 - |z|^2)^{1-\alpha} \|f'_j(z)\| + \frac{\epsilon}{2}, \quad (z \in \mathbb{D}).$$

For each $1 \leq j \leq n$, there exists $r_j \in (0, 1)$ such that

$$\sup_{r_j < |z| < 1} (1 - |z|^2)^{1-\alpha} \|f'_j(z)\| < \frac{\epsilon}{2}.$$

Setting $r = \max\{r_1, \dots, r_n\}$ yields the assertion

$$\sup_{r < |z| < 1} (1 - |z|^2)^{1-\alpha} \|f'(z)\| \leq \epsilon, \quad (f \in K)$$

and (iii) holds.

For the converse, let $\{f_n\}$ be a bounded sequence in $\text{lip}_A(\overline{\mathbb{D}}, X, \alpha)$. Hence, $\{f_n\}$ is an equicontinuous sequence in $C(\overline{\mathbb{D}}, X)$. From (ii) and by generalized Arzela-Ascoli theorem, [3, Theorem A], $\{f_n\}$ is relatively compact in $C(\overline{\mathbb{D}}, X)$. Thus there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is Cauchy in $C(\overline{\mathbb{D}}, X)$. Let $g_n = f_n|_{\mathbb{D}}$. We show that $\{g_{n_k}\}$ is a Cauchy sequence in $\Lambda_\alpha^0(X)$. Fix $\epsilon > 0$. By (iii), there exists $0 < r < 1$ such that for every j ,

$$(1 - |z|^2)^{1-\alpha} \|g'_{n_j}(z)\| < \frac{\epsilon}{4}, \quad (r < |z| < 1).$$

It is easy to check that $\{g'_{n_k}\}$ is Cauchy with respect to compact open topology on $C(\mathbb{D}, X)$. Thus for k and l sufficiently large, we have

$$\sup_{|z| \leq r} \|g'_{n_k}(z) - g'_{n_l}(z)\| < \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\alpha} \|g'_{n_k}(z) - g'_{n_l}(z)\| &\leq \sup_{|z| \leq r} (1 - |z|^2)^{1-\alpha} \|g'_{n_k}(z) - g'_{n_l}(z)\| \\ &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{1-\alpha} \|g'_{n_k}(z) - g'_{n_l}(z)\| \\ &< \epsilon, \end{aligned}$$

which implies that $\{g_{n_k}\}$ is a Cauchy sequence in $\Lambda_\alpha^0(X)$. Thus $\{f_n\}$ is Cauchy in $\text{lip}_A(\overline{\mathbb{D}}, X, \alpha)$, as desired. \square

Regarding the arguments following the proof of Proposition 2.1, for every $\psi \in \Lambda_\beta(K(X, Y))$ there exists an operator-valued function $\Psi \in \text{Lip}_A(\overline{\mathbb{D}}, L(X, Y), \beta)$ such that $\Psi|_{\mathbb{D}} = \psi$. More precisely, for every $z \in \partial\mathbb{D}$, $\Psi_z = \lim_{n \rightarrow \infty} \psi_{z_n}$, where $\{z_n\}$ is any sequence in \mathbb{D} converging to z . Since for every n , $\psi_{z_n} : X \rightarrow Y$ is compact, we deduce that Ψ_z is a compact linear operator from X into Y and $\Psi \in \text{Lip}_A(\overline{\mathbb{D}}, K(X, Y), \beta)$.

Now we can characterize the compact weighted composition operators from $\Lambda_\alpha^0(X)$ into $\Lambda_\beta^0(Y)$.

Theorem 3.3. *Let $W_{\psi, \varphi} : \Lambda_\alpha^0(X) \rightarrow \Lambda_\beta^0(Y)$ be a bounded weighted composition operator.*

(i) *If $W_{\psi, \varphi}$ is compact, then $\psi \in \Lambda_\beta(K(X, Y))$ and*

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| = 0. \quad (3.1)$$

(ii) *If $\psi \in \Lambda_\beta^0(K(X, Y))$ and (3.1) holds, then $W_{\psi, \varphi}$ is compact.*

Proof. (i) Let U_α be the closed unit ball of $\Lambda_\alpha^0(X)$. Since $W_{\psi, \varphi}$ is compact, $W_{\psi, \varphi}(U_\alpha)$ is relatively compact in $\Lambda_\beta^0(Y)$ and hence, $E(W_{\psi, \varphi}(U_\alpha))$ is relatively compact in $\text{lip}_A(\overline{\mathbb{D}}, Y, \beta)$. By Lemma 3.2, we have

$$\lim_{|z| \rightarrow 1^-} \sup_{f \in U_\alpha} (1 - |z|^2)^{1-\beta} \|(W_{\psi, \varphi}(f))'(z)\| = 0.$$

Given $\epsilon > 0$, there exists $r \in (0, 1)$ such that

$$\sup_{r < |z| < 1} \sup_{f \in U_\alpha} (1 - |z|^2)^{1-\beta} \|(W_{\psi, \varphi}(f))'(z)\| < \frac{\epsilon}{3}. \quad (3.2)$$

For every $x \in X$ and every nonzero $a \in \mathbb{D}$, define

$$f_{a,x}(z) = \frac{1}{a} \left(\frac{1 - |a|^2}{(1 - \bar{a}z)^{1-\alpha}} - (1 - \bar{a}z)^\alpha \right) x.$$

One can see that $\{f_{a,x} : 0 \neq a, \|x\| \leq 1\}$ is a bounded subset of $\Lambda_\alpha^0(X)$. Moreover, $f_{a,x}(a) = 0$, $f'_{a,x}(a) = \frac{x}{(1 - |a|^2)^{1-\alpha}}$ and $\sup_{\substack{\|x\| \leq 1 \\ z \in \mathbb{D}}} \|f_{a,x}\|_{\Lambda_\alpha(X)} \leq 3$. Since

$(W_{\psi, \varphi}(f_{\varphi(z), x}))'(z) = \frac{\varphi'(z)}{(1 - |\varphi(z)|^2)^{1-\alpha}} \psi_z(x)$, by relation (3.2) we have

$$\begin{aligned} \sup_{r < |z| < 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| &= \sup_{r < |z| < 1} \sup_{\|x\| \leq 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z x\| \\ &= \sup_{r < |z| < 1} \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|(W_{\psi, \varphi}(f_{\varphi(z), x}))'(z)\| \\ &< \epsilon. \end{aligned}$$

(ii) Let $\psi \in \Lambda_\beta^0(K(X, Y))$ and (3.1) holds. Fixing $\epsilon > 0$, for some $r \in (0, 1)$ we have

$$\sup_{r < |z| < 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| < \frac{\epsilon}{8},$$

and

$$\sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \|\psi'_z\| < \frac{\alpha\epsilon}{8}.$$

Let $\{f_n\}$ be a bounded sequence in $\Lambda_\alpha^0(X)$ such that $\|f_n\|_{\Lambda_\alpha(X)} \leq 1$. Then $\{E(f_n)\}$ is a bounded sequence in $\text{lip}_A(\overline{\mathbb{D}}, X, \alpha)$ and $\{W_{\Psi, \phi}(E(f_n))\}$ is a bounded sequence in $\text{lip}_A(\overline{\mathbb{D}}, Y, \beta)$, which implies that $\{W_{\Psi, \phi}(E(f_n))\}$ is equicontinuous. For every $z \in \overline{\mathbb{D}}$, $\{E(f_n)(\phi(z))\}$ is a bounded sequence in X and $\Psi_z : X \rightarrow Y$ is a compact operator. Hence, $\{W_{\Psi, \phi}(E(f_n))(z)\}$ is relatively compact in Y . From Arzela-Ascoli theorem we deduce that $\{W_{\Psi, \phi}(E(f_n))\}$ is relatively compact in $C(\overline{\mathbb{D}}, Y)$. Therefore, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{W_{\psi, \varphi}(f_{n_k})\}$ is uniformly convergent on compact subsets of \mathbb{D} . Using Cauchy's integral theorem, one can show that $\{(W_{\psi, \varphi}(f_{n_k}))'\}$ is convergent with respect to the compact open topology. Thus for k and l sufficiently large,

$$\sup_{|z| \leq r} \|(W_{\psi, \varphi}(f_{n_k}))'(z) - (W_{\psi, \varphi}(f_{n_l}))'(z)\| < \frac{\epsilon}{4}.$$

Therefore,

$$\begin{aligned} & \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|(W_{\psi, \varphi}(f_{n_k}))'(z) - (W_{\psi, \varphi}(f_{n_l}))'(z)\| \\ & \leq \sup_{|z| \leq r} (1 - |z|^2)^{1-\beta} \|(W_{\psi, \varphi}(f_{n_k}))'(z) - (W_{\psi, \varphi}(f_{n_l}))'(z)\| \\ & \quad + \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_z(f'_{n_k}(\varphi(z)) - f'_{n_l}(\varphi(z)))\| \\ & \quad + \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \|\psi'_z(f_{n_k}(\varphi(z)) - f_{n_l}(\varphi(z)))\| \\ & < \frac{\epsilon}{4} + 2 \sup_{r < |z| < 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| \\ & \quad + \frac{2}{\alpha} \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \|\psi'_z\| < \epsilon. \end{aligned}$$

We conclude that $\{W_{\psi, \varphi}(f_{n_k})\}$ is a Cauchy sequence in $\Lambda_\beta^0(Y)$ and hence $W_{\psi, \varphi}$ is a compact operator. \square

The next corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let $W_{\Psi, \phi} : \text{lip}_A(\overline{\mathbb{D}}, X, \alpha) \rightarrow \text{lip}_A(\overline{\mathbb{D}}, Y, \beta)$ be a bounded weighted composition operator.*

(i) *If $W_{\Psi, \phi}$ is compact, then $\Psi \in \text{Lip}_A(\overline{\mathbb{D}}, K(X, Y), \beta)$ and*

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \|\psi_z\| = 0. \quad (3.3)$$

(ii) *If $\Psi \in \text{lip}_A(\overline{\mathbb{D}}, K(X, Y), \beta)$ and (3.3) holds, then $W_{\Psi, \phi}$ is compact.*

REFERENCES

1. F. Behrouzi and H. Mahyar, *Compact endomorphisms of certain analytic Lipschitz algebras*, Bull. Belg. Math. Soc. **12** (2005), no. 2, 301-312.
2. J. Bonet, P. Domański and M. Lindström, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 233-248.
3. J. T. Chan, *Operators with the disjoint support property*, J. Operator theory **24** (1990), 383-391.
4. N. Dinculeanu, *Vector measures*, Hochschulbücher für Mathematik, Band **64**, VEB Deutscher Verlag Wissenschaften, Berlin, (1966).
5. K. Esmaeili and H. Mahyar, *Weighted composition operators between vector-valued Lipschitz function spaces*, Banach J. Math. Anal. **7** (2013), no. 1, 59-72.
6. J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147-169.
7. S. Ohno, K. Stroethoff, R. Zhao, *Weighted composition operators between Bloch-type spaces* Rocky Mount. J. Math. **33** (2003), 191-215.
8. J. Laitila, *Composition operators and vector-valued BMOA*, Integral Equations Operator Theory **58** (2007), 487-502.
9. J. Laitila, *Weakly compact composition operators on vector-valued BMOA*, J. Math. Anal. Appl. **308** (2005), 730-745.
10. J. Laitila and H.-O. Tylli, *Operator-weighted composition operators on vector-valued analytic function spaces*, Illinois J. Math. **53** (2009), no. 4, 1019-1032.
11. J. Laitila, H.-O. Tylli and M. Wang, *Composition operators from weak to strong spaces of vector-valued analytic functions*, J. Operator Theory **62** (2009), no. 2, 281-295.
12. P. Liu, E. Saksman and H.-O. Tylli, *Small composition operators on analytic vector-valued function spaces*, Pacific J. Math. **184** (1998), 295-309.
13. H. Mahyar and A. H. Sanatpour, *Compact composition operators on certain analytic Lipschitz spaces*, Bull. Iranian Math. Soc. **38** (2012), no. 3, 87-101.

¹DEPARTMENT OF ENGINEERING, ARDAKAN UNIVERSITY, P. O. BOX 89518-95491, ARDAKAN, YAZD, IRAN.

E-mail address: esmaeili@ardakan.ac.ir; k.esmaili@gmail.com